

RAGHAVAN NARSIMHAN'S PROOF OF L. SCHWARTZ'S PERTURBATION THEOREM

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ABSTRACT. Raghavan Narasimhan outlined a new proof of L. Schwartz's perturbation theorem during a course of lectures at IMSc, Chennai in Spring 2007. The details are given.

Professor Raghavan Narasimhan gave a course of lectures on the structure of pseudo-convex manifolds at IMSc, Chennai in Spring 2007. During this course, a new proof of L. Schwartz's perturbation theorem for operators on Fréchet spaces was outlined. The proof is easy in the setting of Hilbert spaces, but for Fréchet spaces (or even Banach spaces) the proofs in the literature are quite hard (see e.g. [1]). However, the Fréchet space version is the one which is needed to give easy proofs of the finite dimensionality of various cohomology groups which occur in complex analysis (in particular, the easiest proof that there exists a meromorphic function on any compact Riemann surface). The proof given here was discovered by Narasimhan shortly after the second edition of his book [1] went to press. Sadly, Professor Narasimhan passed away on October 3, 2015, before a third edition could be brought out.

There are two main ideas in the proof. The first is that a compact set (and hence a compact operator) is small modulo a finite dimensional subspace. The second is that a small perturbation of an onto map is still onto (this is the content of Lemma 3). In the setting of Hilbert spaces, this was proved by C. Neumann in the 19th century using a geometric series argument. Lemma 3 is a delicate adaptation of his argument, which took over a 100 years to discover!

Theorem 1. *Let E, F be Fréchet spaces, and $f, g : E \rightarrow F$ continuous linear maps such that g is onto, and there exists a neighborhood U of 0 in E such that $\overline{f(U)}$ is compact. Then $f + g$ has closed image of finite codimension in F .*

First of all, observe that it suffices to prove that $(f + g)(E)$ has finite codimension in F , because of the following simple lemma.

Lemma 2. *Let E, F be Fréchet spaces and $h : E \rightarrow F$ a continuous linear map. If $h(E)$ has finite codimension in F then $h(E)$ is closed.*

Proof. Let F' be a complement for $h(E)$ in F . Since F' is finite dimensional, it is Fréchet. Let $E' = E/\text{Ker}(h)$. Then $E' \oplus F'$ is Fréchet. Let $\Pi_{F'} : E' \oplus F' \rightarrow F'$ denote the

projection. Then $\Pi_{F'}$ is continuous. Let $h' : E' \rightarrow F$ be the map induced by h . Define $H : E' \oplus F' \rightarrow F$ by $H(x, y) = h'(x) + y$. Then H is continuous, one-one and onto. By the open mapping theorem, H^{-1} is continuous. Moreover, $h(E) = \text{Ker}(\Pi_{F'} \circ H^{-1})$. Therefore $h(E)$ is closed. \square

Since $K = \overline{f(U)}$ is compact and $V = g(U)$ is open (by the open mapping theorem), there exist $y_1, \dots, y_n \in K$ such that $K \subseteq \bigcup_{j=1}^n (y_j + \frac{1}{2}V)$. Put $F' = \text{span}\{y_1, \dots, y_n\}$ and let $f', g' : E \rightarrow F/F'$ be the induced maps. Then g' is onto, $\overline{f'(U)}$ is compact and $\overline{f'(U)} \subseteq \frac{1}{2}g'(U)$. Thus we are reduced to proving the following lemma, which is the heart of the matter.

Lemma 3. *Let E, F be Fréchet spaces, and $f, g : E \rightarrow F$ continuous linear maps. Assume g is onto, and there exists an open symmetric neighborhood U of 0 in E such that $\overline{f(U)}$ is compact and $\overline{f(U)} \subseteq \frac{1}{2}g(U)$. Then $h = f + g$ is also onto.*

Proof. Put $K = \overline{f(U)}$ and $V = g(U)$. By the open mapping theorem, V is open. It suffices to show that h is onto V .

Let $\{W_p\}_{p=1}^\infty$ be a fundamental system of neighborhoods of 0 in E such that each W_p is open, convex and symmetric. From the compactness of K , it follows that for each p , there exists n_p such that $K \subseteq \frac{1}{2}g(U \cap 2^{n_p}W_p) = g(\frac{1}{2}U \cap 2^{n_p-1}W_p)$. Discard some of the W_p and reindex them so that

$$(1) \quad W_{p+1} \subseteq \frac{1}{2}W_p$$

for all p . The new collection is still a fundamental system of neighborhoods of 0.

Let $y_0 \in V$. Then there exists $x_0 \in U$ such that $y_0 = g(x_0)$. Therefore

$$y_1 := y_0 - h(x_0) = -f(x_0) \in K = K \cap \frac{1}{2}V.$$

Therefore there exists $x_1 \in \frac{1}{2}U$ such that $y_1 = g(x_1)$. Therefore

$$y_2 := y_1 - h(x_1) = -f(x_1) \in \frac{1}{2}K = \frac{1}{2}K \cap \frac{1}{4}V.$$

Therefore there exists $x_2 \in \frac{1}{4}U$ such that $y_2 = g(x_2)$. Continuing in this way, we obtain sequences $\{y_j\}_{j=0}^{n_1}$ and $\{x_j\}_{j=0}^{n_1}$ such that

$$\begin{aligned} y_{j+1} &= y_j - h(x_j), \\ y_j &= g(x_j), \\ y_j &\in 2^{-j+1}K \cap 2^{-j}V, \\ x_j &\in 2^{-j}U \end{aligned}$$

Therefore

$$y_{n_1+1} := y_{n_1} - h(x_{n_1}) = -f(x_{n_1}) \in 2^{-n_1}K = 2^{-n_1}K \cap 2^{-n_1-1}V.$$

Therefore there exists $x_{n_1+1} \in 2^{-n_1-1}U \cap \frac{1}{2}W_1$ such that $g(x_{n_1+1}) = y_{n_1+1}$. Continuing in this way, we obtain sequences $\{y_j\}_{j=n_1+1}^{n_2}$ and $\{x_j\}_{j=n_1+1}^{n_2}$ such that

$$\begin{aligned} y_{j+1} &= y_j - h(x_j), \\ y_j &= g(x_j), \\ y_j &\in 2^{-j+1}K \cap 2^{-j}V, \\ x_j &\in 2^{-j}U \cap 2^{n_1-j}W_1 \end{aligned}$$

This whole procedure can be further iterated to obtain sequences $\{y_j\}_{j=0}^{\infty}$ and $\{x_j\}_{j=0}^{\infty}$ such that

$$\begin{aligned} y_{j+1} &= y_j - h(x_j), \\ y_j &= g(x_j), \\ y_j &\in 2^{-j+1}K \cap 2^{-j}V, \\ x_j &\in 2^{-j}U \cap 2^{n_p-j}W_p, \quad \text{if } j > n_p, \quad p = 1, 2, \dots \end{aligned}$$

Observe that if $n_p < k < l \leq n_{p+1}$ then

$$\begin{aligned} x_k + \dots + x_l &\in 2^{n_p-k}W_p + \dots + 2^{n_p-l}W_p \\ &\subseteq (2^{n_p-k} + \dots + 2^{n_p-l})W_p \quad (\text{because } W_p \text{ is convex}) \\ &\subseteq W_p. \end{aligned}$$

So if $n_p < k < l \leq n_q$ then

$$\begin{aligned} x_k + \dots + x_l &\in W_p + \dots + W_{q-1} \\ &\subseteq W_p + \dots + 2^{p-q}W_p \quad (\text{by (1)}) \\ &\subseteq (1 + \dots + 2^{p-q})W_p \quad (\text{because } W_p \text{ is convex}) \\ &\subseteq 2W_p. \end{aligned}$$

Therefore $z_j := x_0 + x_1 + \dots + x_j$ is Cauchy. Let $z = \lim_{j \rightarrow \infty} z_j$. Then

$$\begin{aligned} y_0 - h(z) &= \lim_{j \rightarrow \infty} (y_0 - h(z_j)) \\ &= \lim_{j \rightarrow \infty} (y_j - h(x_j)) \\ &= \lim_{j \rightarrow \infty} y_{j+1} \\ &= \lim_{j \rightarrow \infty} g(x_{j+1}) \\ &= 0. \end{aligned}$$

Therefore h is onto V . □

REFERENCES

1. Raghavan Narasimhan and Yves Nievergelt, *Complex analysis in one variable*, second ed., Birkhäuser Boston Inc., Boston, MA, 2001. MR MR1803086 (2002e:30001)

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